

A SIMPLE E_8 CONSTRUCTION

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ABSTRACT. In this short letter we conclude our program, started in [6], of building up explicit generalized Euler angle parametrizations for all exceptional compact Lie groups. In this last step we solve the problem for E_8 .

1. INTRODUCTION

In [6], we started a project having the aim of providing an explicit realization of all compact (simply connected) exceptional Lie groups, including an explicit determination of the range of parameters in order to cover the group exactly one time. The problem of specifying the range is indeed a complication in several numerical simulations involving global properties of the group and is really time consuming if it has to be treated numerically. To this end, we have introduced the generalized Euler parametrization, which has the advantage of characterizing the complete ranges in a very simple form (see [5] for a review). We have done this for all the exceptional Lie groups but E_8 in [4], [2], [3], and [7]. Here we complete our program by providing a generalized Euler parametrization of E_8 starting from its maximal compact subgroup $S_8(16) \equiv \text{Spin}(16)/\mathbb{Z}_2$. We will employ a general method developed in [8], and refer, for the details proving our assertions, to that paper.

It is worth to mention that, beyond the applications in string theories, GUT theories and theories of everything [10, 11], the exceptional Lie group E_8 finds applications also in more phenomenological areas of physics like, for example, the quasi-one-dimensional Ising ferromagnet CoNb_2O_6 described in [9].

2. THE ALGEBRA

A very simple construction of the E_8 Lie algebra is given in [1], chapter 6 (see also chapter 7). The compact real form is obtained by starting from the Lie algebra $L := \text{Lie}(\text{Spin}(16))$ (or, equivalently by its adjoint irrep. **120**) and its real irreducible representation $\Delta_+ \equiv \mathbf{128}$. Let us review how this construction works concretely.

The $\text{Spin}(16)$ subalgebra can be thought of as generated by elements $J_{ij} = -J_{ji}$, $1 \leq i < j \leq 16$ with commutators

$$[J_{ij}, J_{kl}] = \delta_{jk}J_{il} - \delta_{jl}J_{ik} - \delta_{ik}J_{jl} + \delta_{il}J_{jk} \equiv \sum_{m < n} C_{ij,kl}{}^{mn} J_{mn}. \quad (2.1)$$

The support space for Δ_+ has generators Q_α , $\alpha = 1, \dots, 128$, transforming as a Majorana-Weyl spinors of $\text{Spin}(16)$.¹ Let γ_j , $j = 1, \dots, 16$ be the 128×128 gamma matrices generating the corresponding Clifford algebra. Thus, the representation Δ_+ has generators

$$\Delta_{ij} := \Delta_+(J_{ij}) = \frac{1}{4}[\gamma_i, \gamma_j]. \quad (2.2)$$

The adjoint representation **248** of the E_8 algebra is then completely specified by adding to the commutators rules (2.1) the remaining commutators

$$\begin{aligned} [J_{ij}, Q_\alpha] &= \frac{1}{4} \sum_{\beta} [\gamma_i, \gamma_j]_{\alpha\beta} Q_\beta \\ [Q_\alpha, Q_\beta] &= \frac{1}{8} \sum_{i,j} [\gamma^i, \gamma^j]_{\alpha\beta} J_{ij}. \end{aligned}$$

In this way we get the adjoint representation ρ with generators $\rho(J_{ij}), \rho(Q_\alpha)$, $1 \leq i < j \leq 16$, $\alpha = 1, \dots, 128$ given by

$$M_{ij} := \rho(J_{ij}) = \begin{pmatrix} (M_{ij})_{kl,mn} & 0 \\ 0 & (M_{ij})_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} C_{ij,kl}{}^{mn} & 0 \\ 0 & (\Delta_{ij})_{\alpha\beta} \end{pmatrix}, \quad 1 \leq i < j \leq 16, \quad (2.3)$$

$$M_\alpha := \rho(Q_\alpha) = \begin{pmatrix} 0 & (M_\alpha)_{kl,\beta} \\ (M_\alpha)_{\gamma,mn} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4(\Delta^{kl})_{\alpha\beta} \\ -4(\Delta^{mn})_{\alpha\gamma} & 0 \end{pmatrix}, \quad \alpha = 1, \dots, 128. \quad (2.4)$$

A realization of this matrices is given by the Mathematica program on <http://www.dfm.uninsubria.it/E8/>, useful for the explicit computations. In particular, using that program one can check for example that a possible choice of a Cartan subalgebra is given by the matrices $\rho(Q_\alpha)$ with $\alpha = 1, 8, 26, 31, 43, 46, 52, 53$. We rename these matrices C_a , $a = 1, \dots, 8$.

¹These can be easily obtained by the very useful Mathematica package `gamma.m` by Jeremy Michelson, available on <http://www.physics.ohio-state.edu/~jeremy/>

3. THE GROUP

We can now construct the group E_8 by specializing the Euler parametrization and specifying the range of parameters. As shown in [1], chapter 7, the maximal compact subgroup corresponding to the $SO(16)$ subalgebra is a $Spin(16)/\mathbb{Z}_2$ group. More specifically (see [12]) there are three possible non isomorphic quotients of this kind, which are the SO group $SO(16)$ and two “semispin” groups denoted $S_s(16)$ and $S'_s(16)$. The quotient realized here is the semispin group $S:=S_s(16)$ [12].

Let $S[x_1, \dots, x_{120}]$ any parametrization of S , for example an Euler parametrization. This is standard and we will not discuss it any further here. With this at hand, a parametrization of E_8 is given by

$$E_8[x_1, \dots, x_{120}; y_1, \dots, y_8; z_1, \dots, z_{120}] = (S[x_1, \dots, x_{120}]/\mathbb{Z}_2^8) e^{\sum_{a=1}^8 y^a C_a} S[z_1, \dots, z_{120}]. \quad (3.1)$$

Here, the range for the z coordinates must be chosen such to cover S exactly one times (up to a subset of zero measure). The action of the \mathbb{Z}_2^8 finite group in the left factor eliminates redundances and is realized simply by restricting the range of the parameters x (w.r.t. the ranges of the z_i). For example, in the generalized Euler construction of S , all parameters can be chosen periodic, but only 8 parameters cover the whole period in the definition of the ranges. In this case, the action of the finite group just reduces these ranges to half the period.

In this way it remains to specify the range for the y coordinates. By choosing y_a in a period one gets a quite large redundancy, so that a strategy to reduce the ranges is necessary. A general solution of this problem is given in [8], and works as follows. Let $\alpha^1, \dots, \alpha^8$ be any given choice of simple roots with components $(\alpha_1^i, \dots, \alpha_8^i)$ w.r.t. the basis C_1, \dots, C_8 of the Cartan subalgebra. Moreover, let $\alpha^L = \sum_{a=1}^8 n_a \alpha^a$ the longest root. For E_8 , $(n_1, \dots, n_8) = (2, 3, 4, 6, 5, 4, 3, 2)$. Thus, the (almost everywhere) injective range for the parameters y^a is

$$0 \leq \sum_{a=1}^8 \alpha_a^i y^a < \pi, \quad i = 1, \dots, 8, \quad (3.2)$$

$$0 \leq \sum_{a=1}^8 \alpha_a^L y^a < \pi. \quad (3.3)$$

In our case we get that a possible choice of simple roots is

$$\alpha^1 = \frac{1}{2}(1, -1, -1, -1, -1, -1, -1, 1), \quad (3.4)$$

$$\alpha^2 = (1, 1, 0, 0, 0, 0, 0, 0), \quad (3.5)$$

$$\alpha^3 = (0, -1, 1, 0, 0, 0, 0, 0), \quad (3.6)$$

$$\alpha^4 = (0, 0, -1, 1, 0, 0, 0, 0), \quad (3.7)$$

$$\alpha^5 = (0, 0, 0, -1, 1, 0, 0, 0), \quad (3.8)$$

$$\alpha^6 = (0, 0, 0, 0, -1, 1, 0, 0), \quad (3.9)$$

$$\alpha^7 = (0, 0, 0, 0, 0, -1, 1, 0), \quad (3.10)$$

$$\alpha^8 = (0, 0, 0, 0, 0, 0, -1, 1),$$

so that the longest root is $\alpha^L = (0, 0, 0, 0, 0, 0, 1, 1)$. Then, the range of the parameters is specified by

$$\begin{aligned} 0 &\leq \frac{1}{2}(y^1 - y^2 - y^3 - y^4 - y^5 - y^6 - y^7 + y^8) < \pi \\ 0 &\leq y^1 + y^2 < \pi \\ 0 &\leq y^2 - y^1 < \pi \\ 0 &\leq y^3 - y^2 < \pi \\ 0 &\leq y^4 - y^3 < \pi \\ 0 &\leq y^5 - y^4 < \pi \\ 0 &\leq y^6 - y^5 < \pi \\ 0 &\leq y^7 - y^6 < \pi \\ 0 &\leq y^7 + y^8 < \pi, \end{aligned}$$

or equivalently

$$\begin{aligned}
0 &\leq y^1 < \frac{\pi}{6} \\
y^1 &\leq y^2 < \frac{\pi + y^1}{7} \\
y^2 &\leq y^3 < \frac{1}{6}(\pi + y^1 - y^2) \\
y^3 &\leq y^4 < \frac{1}{5}(\pi + y^1 - y^2 - y^3) \\
y^4 &\leq y^5 < \frac{1}{4}(\pi + y^1 - y^2 - y^3 - y^4) \\
y^5 &\leq y^6 < \frac{1}{3}(\pi + y^1 - y^2 - y^3 - y^4 - y^5) \\
y^6 &\leq y^7 < \frac{1}{2}(\pi + y^1 - y^2 - y^3 - y^4 - y^5 - y^6) \\
-y^1 + y^2 + y^3 + y^4 + y^5 + y^6 + y^7 &\leq y^8 < \pi - y^7.
\end{aligned}$$

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